

CUSP FORMATION FOR A NONLOCAL EVOLUTION EQUATION

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ABSTRACT. In this paper, we introduce a nonlocal evolution equation inspired by the Córdoba-Córdoba-Fontelos nonlocal transport equation. The Córdoba-Córdoba-Fontelos equation can be regarded as a model for the 2D surface quasigeostrophic equation or the Birkhoff-Rott equation. We prove blowup in finite time, and more importantly, investigate conditions under which the solution forms a cusp in finite time.

1. INTRODUCTION

The non-linear, non-local active scalar equation

$$(1) \quad \theta_t + u\theta_y = 0, \quad u = H\theta$$

for $\theta = \theta(y, t)$ was proposed by A. Córdoba, D. Córdoba and A. Fontelos [7] as a one-dimensional analogue of the 2D surface quasigeostrophic equation (SQG) [6] and the Birkhoff-Rott equation [15]. We refer to (1) as the CCF equation.

The study of one-dimensional equations modeling various aspects of three- and two-dimensional fluid mechanics problems has a long-standing tradition ([1, 5, 9, 11]).

It is well-known that smooth solutions of (1) lose their regularity in finite time ([7, 13, 16]). However, little is understood about the precise way in which the singularity forms. A particularly simple scenario is as follows: We consider smooth, even solutions, i.e. $\theta(-y, t) = \theta(y, t)$ such that the initial condition $\theta_0(y)$ has a single nondegenerate maximum and decays sufficiently quickly at infinity. Numerical simulations [7, 16] seem to indicate that the solution forms a *cusp* at the singular time T_s , so that

$$(2) \quad \theta(y, T_s) \sim \theta_0(0) - \text{const.}|y|^{1/2}$$

close to the origin. In [16], the authors make the conjecture that in a generic sense, all maxima eventually develop into cusps of the form (2).

Besides the original blowup proof [7], various proofs of blowup for (1) have been found recently [16]. In [8], a discrete model for (1) was studied.

Many of the known proofs work when an additional viscosity term is present in (1). On the other hand none of them explains the cusp formation since the shape of the solution is not controlled at the singular time. The task of establishing that solutions of (1) exhibit cusp formation appears to be challenging.

We believe that making progress on the question of cusp formation is important for the following reasons. In order to prove cusp formation for (1), we need to develop insight into

the intrinsic mechanism of singularity formation. This mechanism is not well-understood and seems to favor blowup at certain predetermined points. Moreover, the solution apparently remains regular outside of these points - even at the time of blowup.

This situation is similar to recent in numerical observations for the three-dimensional Euler equations by T. Hou and G. Luo [9]. There, the authors exhibit solutions for which the magnitude of the vorticity vector appears to become infinite at an intersection point of a domain boundary with a symmetry axis. Their blowup scenario is referred to as the *hyperbolic flow scenario*. In [10], a 1D model problem for axisymmetric flow was introduced, and finite time blowup was proven in [2]. We would also like to mention [12], in which the authors study a simplified model (proposed in [3]) of the equation introduced by T. Hou and G. Luo in [10], proving the existence of self-similar solutions using computer-assisted means.

A sufficiently detailed understanding of the singularity formation mechanism for nonlocal active scalar equations like (1) is very likely a prerequisite for obtaining a blowup proof for the 3D Euler equations. We refer to [4] for a thorough discussion of this outstanding and challenging problem.

In this paper, our goal is to lay groundwork for future investigation by studying a nonlocal active scalar equation for which the singular behavior of the solution at the blowup time can be characterized. We develop new techniques that allow us to obtain some control on the singular shape. Our model problem is related to (1) and reads as follows:

$$(3) \quad \begin{aligned} \theta_t + u\theta_y &= 0, \\ u_y(y, t) &= \int_y^\infty \frac{\theta_y(z, t)}{z} dz, \quad u(0, t) = 0. \end{aligned}$$

We consider solutions θ defined on $[0, \infty)$ and think of them as being extended to \mathbb{R} as even functions. Our main result, Theorem 2, states that solutions blow up in finite time, either forming a cusp or a needle-like singularity. To the best of our knowledge, this is the first time such a scenario has been rigorously established.

Our paper is structured as follows: in the remaining part of this introductory section, we motivate the introduction of (3). In section 2, we state our main results. The remainder of our paper provides the proofs of Theorems 1 and 2.

1.1. Derivation of the model equation. An essential issue with nonlocal transport equations is predicting from the outset where the singularity will form. The even symmetry of θ helps, by creating a stagnant point of the flow at the origin. At this stage of our understanding of blowup scenarios, however, this still does not rule out the possibility of a singularity also forming somewhere else. Our goal is to simplify by writing down a model where singularities can only form at a given point, following ideas of “hyperbolic approximation” or “hyperbolic cut-off” from [14] and [3]. We first describe how this applies to (1).

The velocity gradient of equation (1) is given by $u_y = H\theta_y$. Using the odd symmetry of θ_y , we can write

$$u_y(y, t) = PV \int_0^{2y} \frac{2z\theta_y(z, t) dz}{z^2 - y^2} + \int_{2y}^\infty \frac{2z\theta_y(z, t) dz}{z^2 - y^2}.$$

In a first approximation we only retain the long-range part of the interaction, and therefore consider the model Biot-Savart law

$$(4) \quad u_y(y, t) = \int_y^\infty \frac{\theta_y(z, t) dz}{z}, \quad u(0, t) = 0.$$

The long-range part of the interaction has been approximated by shifting the kernel singularity to the origin (and we have also dropped the non-essential factor 2). In general, the hyperbolic cut-off emphasizes the non-local role of the fluid surrounding the singular point as a leading part of the blowup mechanism. Due to certain monotonicity properties, the intrinsic blowup mechanism becomes transparent (see Remark 1).

Our main inspiration to derive (3) came from [14], where the hyperbolic flow scenario for the 2D Euler equation on a disc was considered. There, a useful new representation of the velocity field for the 2D Euler equation was discovered. Briefly, the velocity field close to a stagnant point of the flow at the intersection of a domain boundary and a symmetry axis of the solution was decomposed into a main part and error term

$$u_1(x_2, x_2, t) = -x_1 \iint_{y_1 \geq x_1, y_2 \geq x_2} \frac{y_1 y_2}{|y|^4} \omega(y, t) dy_1 dy_2 + x_1 B_1(x_1, x_2, t).$$

The error term B_1 could be controlled in a certain sector. Note that, as in (4), the main part of the velocity field is given by an integral over a kernel with singularity at $y = (0, 0)$.

Finally we note that K. Choi, A. Kiselev and Y. Yao use a similar process in [3] to approximate a one-dimensional model of an equation proposed by T. Hou and G. Luo in [10].

2. MAIN RESULTS

We prove that for the model (3), solutions blow up in finite time for generic data, and that the singularity is either a cusp or a needle-like singularity. We are looking for solutions of (3) in the class

$$(5) \quad \theta(y, t) \in C([0, T), C_0^2(\mathbb{R}^+)) \cap C^1([0, T), C^1(\mathbb{R}^+)), \quad \theta_y(0, t) = 0,$$

where $\mathbb{R}^+ = [0, \infty)$ and $C_0^2(\mathbb{R}^+)$ denotes twice continuously differentiable functions with compact support in \mathbb{R}^+ . This is a natural class for solutions of (3) which have a single maximum at the origin.

Our results are as follows:

Theorem 1. *The problem (3) is locally well-posed for compactly supported initial data*

$$\theta_0 \in C_0^2(\mathbb{R}^+), \quad \partial_y \theta_0(0) = 0.$$

Theorem 2. *Assume the initial data θ_0 is compactly supported, nonincreasing, nonnegative and is such that*

$$\partial_{yy} \theta_0(0) < 0.$$

Then there exists a finite time $T_s > 0$ and constants $\nu \in (0, 1), c > 0$ depending on the initial data such that $\theta(y, t) \in C^1([0, T_s), C_0^2(\mathbb{R}^+))$ and

$$\theta(y, T_s) = \lim_{t \rightarrow T_s} \theta(y, t)$$

exists for all $y \in \mathbb{R}^+$ and

$$\theta(\cdot, T_s) \in C^2(0, \infty).$$

Moreover,

$$(6) \quad \theta(y, T_s) \leq \theta_0(0) - c|y|^\nu \quad (y \in \mathbb{R}^+).$$

The singularity formed at time T_s is at least a cusp, but can potentially also be “needle-like” (see Figure 1). Needle formation arises when

$$\lim_{y \rightarrow 0^+} \theta(y, T_s) < \theta_0(0).$$

To the best of our knowledge, this is the first time singularity formation of the type described is rigorously established for a nonlocal active scalar transport equation.

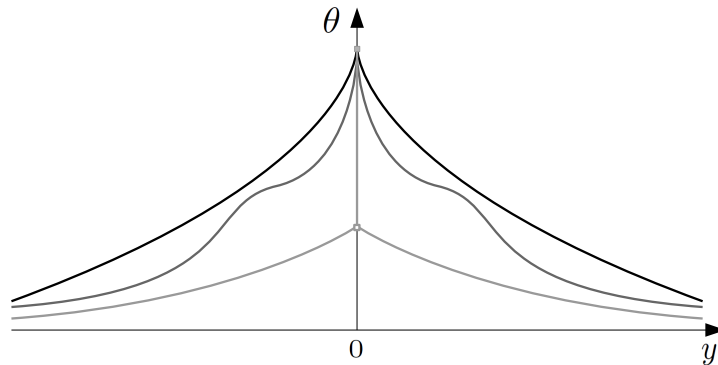


FIGURE 1. Illustration of the upper bound (black), $\theta(\cdot, T_s)$ at singular time with cusp (grey) and a scenario with “needle formation” (light grey). Note that all three have the same value at $y = 0$.

3. PROOFS

3.1. An equation for the flow map. Our approach is characterized by working in Lagrangian coordinates and exploiting the properties of the flow map. From now on, x denotes the Lagrangian space variable. The flow map Φ associated to (3) satisfies

$$(7) \quad \Phi_t(x, t) = u(\Phi(x, t), t).$$

Note that because of $u(0, t) = 0$, $\Phi(0, t) = 0$ holds. The basic equation for the stretching (derivative of the flow map $\phi = \partial_x \Phi$) follows from differentiating (7):

$$(8) \quad \phi_t(x, t) = u_y(\Phi(x, t), t)\phi(x, t).$$

With some nonnegative, compactly supported, smooth $g : [0, \infty) \rightarrow \mathbb{R}$, $g \not\equiv 0$, one can make an ansatz for the solution θ by setting

$$(9) \quad \theta(y, t) = g(\Phi^{-1}(y, t)).$$

The following relation holds:

$$(10) \quad \theta_y(y, t) = g'(\Phi^{-1}(y, t))/\phi(\Phi^{-1}(y, t)).$$

Computing $u_y(\Phi(x, t), t)$ using (4), (10) and the substitution $z = \Phi^{-1}(y, t)$ gives

$$(11) \quad u_y(\Phi(x, t), t) = \int_{\Phi(x, t)}^{\infty} \frac{g'(\Phi^{-1}(y, t))}{y \phi(\Phi^{-1}(y, t))} dy = \int_x^{\infty} \frac{g'(z)}{\Phi(z, t)} dz.$$

Combining this with (8), we obtain the following central evolution equation for ϕ :

$$(12) \quad \begin{aligned} \phi_t(x, t) &= \phi(x, t) \int_x^{\infty} \frac{g'(z)}{\Phi(z, t)} dz, & \Phi(z, t) &= \int_0^z \phi(\sigma, t) d\sigma. \\ \phi(x, 0) &= \phi_0(x) \end{aligned}$$

Here, (ϕ_0, g) are given functions with $(\phi_0, g) \in C^1(\mathbb{R}^+) \times C_0^2(\mathbb{R}^+)$ having the properties

$$(13) \quad \inf_{\mathbb{R}^+} \phi_0 > 0, \quad g'(0) = 0.$$

We shall consider solutions $\phi \in C^1(\mathbb{R}^+ \times [0, T])$ with the additional property

$$(14) \quad \inf_{(x, t) \in \mathbb{R}^+ \times [0, T]} \phi(x, t) > 0.$$

The following Lemma clarifies the relation between θ and (ϕ_0, g) :

Lemma 1. *Let $\theta_0 \in C_0^2(\mathbb{R}^+)$ with $\partial_y \theta_0(0) = 0$ be given. Suppose*

$$\phi \in C^1(\mathbb{R}^+ \times [0, T])$$

is a solution of (12) with given (ϕ_0, g) satisfying (13) and

$$(15) \quad \theta_0(y) = g(\Phi_0^{-1}(y)) \quad (y \in \mathbb{R}^+), \quad \text{where } \Phi_0(x) = \int_0^x \phi_0(\sigma) d\sigma.$$

Then $\theta(y, t) = g(\Phi^{-1}(y, t)) \in C([0, T], C_0^2(\mathbb{R}^+)) \cap C^1([0, T], C^1(\mathbb{R}^+))$ is a solution of (3) as long as (14) holds.

Proof. Observe first that the definition of g in (15) implies $g'(0) = 0$ via the assumption $\partial_y \theta_0(0) = 0$. Let $\theta(y, t)$ be defined by $\theta(y, t) = g(\Phi^{-1}(y, t))$ (note that $\Phi^{-1}(\cdot, t)$ is well-defined for all $t \in [0, T]$ because of (14)). A calculation (see (11)) shows that the integral

$$\int_x^\infty \frac{g'(z)}{\Phi(z, t)} dz$$

equals $u_y(\Phi(x, t), t)$, where u is the velocity field (4). Using (12), this implies

$$\partial_t \partial_x \Phi(x, t) = \partial_x \Phi(x, t) u_y(\Phi(x, t), t).$$

Taking into account $\Phi(0, t) = 0$ and integrating with respect to x , we obtain $\partial_t \Phi(x, t) = u(\Phi(x, t), t)$. By differentiating

$$\theta(\Phi(x, t), t) = g(x)$$

in time we see that θ satisfies equation (3). The initial condition $\theta(y, 0) = \theta_0(y)$ follows from (15). Finally, the required regularity for θ is also straightforward to verify. To check for instance $\theta(\cdot, t) \in C^2(\mathbb{R}^+)$ we observe first that (10) implies that θ_y is continuous on \mathbb{R}^+ . Differentiating (10), we obtain

$$\theta_{yy}(y, t) = \frac{g''(\Phi^{-1}(y, t)) - g'(\Phi^{-1}(y, t))\phi'(\Phi^{-1}(y, t), t)\phi(\Phi^{-1}(y, t), t)^{-1}}{\phi(\Phi^{-1}(y, t), t)^2}$$

and thus $\theta_{yy} \in C(\mathbb{R}^+)$. □

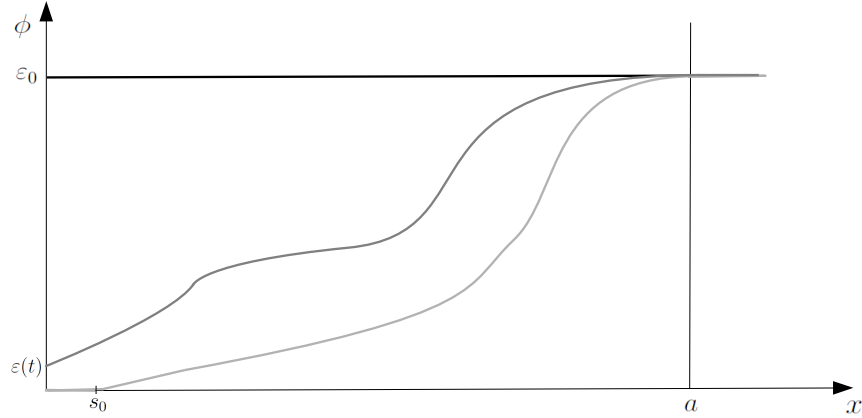


FIGURE 2. Illustration of ϕ .

Remark 1. A fairly clear picture of the blowup mechanism emerges by visualizing the solution of (12). Later, we will choose ϕ_0 to be a positive constant and adjust g so that we obtain solutions of (3) by using Lemma 1. $\phi(x, t)$ is monotone nonincreasing in x and is depicted in Figure 2 for times $t > 0$. The blowup happens when $\phi(0, t)$ reaches zero in finite time. The

intuitive reason is that at the singular time T_s , the odd continuation of the inverse $\Phi^{-1}(\cdot, T_s)$ to $(-\infty, \infty)$ is not differentiable in $x = 0$. The main driving mechanism for blowup in finite time is the behavior of the following integral:

$$-\int_0^\infty \frac{g'(z) dz}{\Phi(z, t)}.$$

We shall show that it will be at least growing like $\phi(0, t)^{-1+\beta}$ with some $\beta \in (0, 1)$, thus implying $\phi_t(0, t) \leq -C\phi(0, t)^\beta$.

The following Theorem shows that nontrivial solutions of (12) cannot be defined on an infinite time interval. We present this Theorem for illustrative purposes, to show that finite-time blowup via a contradiction argument can easily be obtained for (12). The task of characterizing the singular shape is, however, much more challenging (see section 3.3 below).

Theorem 3. *Suppose (13) holds for (ϕ_0, g) and that $g(0) > 0$. There can be no solution of (12) $\phi \in C^1(\mathbb{R}^+ \times [0, \infty))$ satisfying (14).*

Proof. For the purpose of deriving a contradiction, let us assume that $\phi(\cdot, t)$ is defined for $t \in [0, \infty)$. (14) implies $\Phi(x, t) \geq 0$ for all times. Let $[0, a] = \text{supp } g$. By taking the derivative of $\Phi(a, t) = \int_0^a \phi(\sigma, t) d\sigma$ with respect to t and using (12) we get

$$\Phi_t(a, t) = \int_0^a \int_\sigma^a \phi(\sigma, t) \frac{g'(z)}{\Phi(z, t)} dz d\sigma.$$

The integral on the right-hand side can be written as

$$\int_0^a \int_0^z \phi(\sigma, t) \frac{g'(z)}{\Phi(z, t)} d\sigma dz = \int_0^a \Phi(z, t) \frac{g'(z)}{\Phi(z, t)} dz = g(a) - g(0).$$

Because of $g(a) = 0$, $\Phi_t(a, t) = -g(0) < 0$, giving a contradiction for large times. \square

3.2. Local existence.

Theorem 4 (Local existence and uniqueness). *The problem (12) has a unique local solution $\phi \in C^1(\mathbb{R}^+ \times [0, T])$ for some $T > 0$, provided that $\phi_0 \in C^1(\mathbb{R}^+)$, $g \in C_0^2(\mathbb{R}^+)$ and (13) holds for (ϕ_0, g) . Moreover,*

$$\inf_{(x,t) \in \mathbb{R}^+ \times [0, T]} \phi(x, t) > 0.$$

Proof. The proof is standard, so we just sketch a few details. Let $\text{supp } g' = [0, a]$. Rewrite (12) as an integral equation by integrating (12) in time

$$(16) \quad \phi(x, t) = \phi_0(x) + \int_0^t \phi(x, s) \left(\int_x^\infty \frac{g'(z)}{\Phi(z, s)} dz \right) ds =: \phi_0(x) + \mathcal{G}[\phi](x, t).$$

Given (g, ϕ_0) let the set $X_{T, \mu}$ be the set of functions ϕ defined by the following conditions:

$$\phi \in C([0, T], C^1(\mathbb{R}^+)), \quad \|\phi - \phi_0\|_{C([0, T], C^1[0, a])} \leq \mu, \quad \phi(x, t) = \phi_0(x) \text{ for } x \geq a$$

where $T, \mu > 0$ are to be chosen below. The norm $\|\cdot\|_{C([0,T],C^1[0,a])}$ is given by

$$\max_{t \in [0,T]} (\|f(\cdot, t)\|_\infty + \|f_x(\cdot, t)\|_\infty),$$

$\|\cdot\|_\infty$ denoting the supremum norm on $[0, a]$.

First we need to show that the operator $\phi_0 + \mathcal{G}$ is well-defined. For $\phi \in X_{T,\mu}$ we have the following estimate:

$$\begin{aligned} \Phi(z, t) &= \int_0^z \phi(\sigma, t) d\sigma \geq - \left| \int_0^z (\phi(\sigma, t) - \phi_0(\sigma)) d\sigma \right| + \int_0^z \phi_0(\sigma) d\sigma \\ &\geq -\mu z + \left(\min_{\mathbb{R}^+} \phi_0 \right) z = \left(\min_{[0,a]} \phi_0 - \mu \right) z. \end{aligned}$$

So if $\mu = \frac{1}{4} \min_{\mathbb{R}^+} \phi_0$, then we have

$$\frac{g'(z)}{\Phi(z, t)} \leq C(g, \phi_0) \quad (0 \leq z \leq a)$$

because of $g'(0) = 0$, i.e. $g'(z) \leq \text{const.} \cdot z$. Using $\|\partial_x \phi\|_\infty \leq \mu + \|\partial_x \phi_0\|_\infty$, $\|\phi\|_\infty \leq \mu + \|\phi_0\|_\infty$ and (14) we obtain

$$|\mathcal{G}[\phi](x, t)| \leq C(g, \phi_0)T, \quad |\partial_x \mathcal{G}[\phi](x, t)| \leq C(g, \phi_0)T,$$

and $\mathcal{G}[\phi](x, t) = 0$ for all $x \geq a, t \in [0, T]$. Hence $\phi \mapsto \phi_0 + \mathcal{G}[\phi]$ maps $\phi \in X_{T,\mu}$ and into $X_{T,\mu}$ for sufficiently small $T > 0$. A straightforward, but tedious calculation shows the inequality

$$\|\mathcal{G}[\phi] - \mathcal{G}[\psi]\|_{C([0,T],C^1[0,a])} \leq C(\mu, g, a, \phi_0)T \|\phi - \psi\|_{C([0,T],C^1[0,a])}.$$

The proof is concluded by applying the Contraction Mapping Theorem to the operator

$$\phi_0 + \mathcal{G},$$

choosing $T > 0$ to be sufficiently small to ensure the contraction property. Note that $\phi \in X_{T,\mu}$ satisfying the equation (16) lies automatically in $C^1(\mathbb{R}^+ \times [0, T])$. \square

The proof of Theorem 1 follows now from Theorem 4, by taking the local solution of (12) and defining θ via (9). More precisely, we may take e.g. $\phi_0 = 1, g(x) = \theta_0$.

3.3. Proof of Theorem 2. *Setup for the proof and preparatory lemmas.* For convenience, we set

$$\phi_0(x) = \varepsilon_0 > 0,$$

with small $0 < \varepsilon_0 < 1$ to be chosen later, instead of starting with $\phi_0(x) \equiv 1$ (see also Remark 2). To obtain the given initial condition θ_0 we set

$$(17) \quad g(z; \varepsilon_0) := \theta_0(\varepsilon_0 z) \in C^2(\mathbb{R}^+).$$

Note that (17) ensures that θ defined by $\theta(y, t) = g(\Phi^{-1}(y, t); \varepsilon_0)$ satisfies the initial condition $\theta(\cdot, 0) = \theta_0$ for any $\varepsilon_0 > 0$. Observe carefully that variation of ε_0 does not correspond to a rescaling of the initial data for the equation (3).

Now fix some constants $0 < K_0 < K_1$ such that

$$(18) \quad \varepsilon_0^2 K_0 z \leq -g'(z; \varepsilon_0) \leq \varepsilon_0^2 K_1 z \quad (0 \leq z \leq 1)$$

To see that such constants exist, observe that $g''(z; \varepsilon_0) = \varepsilon_0^2 \partial_{yy} \theta_0(\varepsilon_0 z)$ and $g'(0; \varepsilon_0) = 0$; just take K_0 to be slightly lower and K_1 to be slightly larger than $-\partial_{yy} \theta_0(0) > 0$ and choose ε_0 sufficiently small so that (18) holds for $0 \leq z \leq 1$. In the following, we write $g(z; \varepsilon_0) = g(z)$ for convenience.

We write

$$(19) \quad \varepsilon(t) = \phi(0, t).$$

By assumption, $\theta_y(z, 0) < 0$, so that $g' < 0$. Note that (12) implies $\varepsilon_t(t) < 0$ as long as the smooth solution can be continued.

Lemma 2. *The following equation holds for $\eta(x, t) = \phi(x, t)/\varepsilon(t)$:*

$$(20) \quad \eta_t(x, t) = -\eta(x, t) \int_0^x \frac{g'(z) dz}{\Phi(z, t)}.$$

Moreover, $\phi(x, t)$ is monotone nondecreasing in x for fixed t .

Proof. A direct computation gives

$$\eta_t(x, t) = \frac{\varepsilon(t)\phi_t(x, t) - \phi(x, t)\varepsilon_t(t)}{\varepsilon^2(t)} = -\eta(x, t) \int_0^x \frac{g'(z) dz}{\Phi(z, t)}.$$

To see that ϕ is monotone in x , observe that because of $g' < 0$, we have for $x_1 < x_2$

$$\phi_t(x_1, t) \leq \phi(x_1, t) \int_{x_2}^\infty \frac{g'(z) dz}{\Phi(z, t)} \leq \frac{\phi(x_1, t)}{\phi(x_2, t)} \phi_t(x_2, t)$$

and thus $(\log \phi(x_1, t))_t \leq (\log \phi(x_2, t))_t$, and the monotonicity follows since $\phi(x_1, 0) = \phi(x_2, 0)$. \square

Lemma 3. *A solution $\phi(x, t)$ satisfies the bound $\phi(x, t) \geq \varepsilon(t)$ for all $x \in \mathbb{R}^+$. If $T > 0$ is such that $\phi \in C^1(\mathbb{R}^+ \times [0, T))$ and $\lim_{t \rightarrow T} \varepsilon(t) > 0$, then the solution ϕ can be continued to a slightly larger time interval.*

Proof. This follows from Lemma 2. From equation (20) and $\eta(x, 0) = 1$ we get immediately $\eta(x, t) \geq 1$, so that actually $\min_{x \in \mathbb{R}^+} \phi(x, t) = \varepsilon(t)$. Now apply Theorem 4. \square

Let $\beta \in (0, 1)$ and define

$$l(t) = \varepsilon^\beta(t).$$

Let $\kappa > 1$ be such that

$$(21) \quad \varepsilon_0 \leq \kappa \varepsilon_0^{1/\beta}$$

We consider now the following bootstrap or barrier property (B).

$$(B) \quad \phi(x, t) \leq \kappa \varepsilon(t) \text{ for all } x \in [0, l(t)]$$

The continuity of $t \mapsto \phi(\cdot, t)$ implies that (B) holds for some short time interval $[0, \tau)$, $\tau > 0$. We now extend the validity of the bootstrap property, with uniform κ , to the whole existence interval of our solution ϕ by utilizing a kind of *continuous induction*, or *bootstrap argument*, or alternatively speaking a *nonlocal maximum principle*.

- In order to state the argument in a clear fashion, we now let

$$\phi \in C^1([0, T_s) \times \mathbb{R}^+)$$

be the solution of (12) with data $(\varepsilon_0, g(\cdot; \varepsilon_0))$ given by (17), defined on its maximal existence interval $[0, T_s)$. That is, we continue the local solution ϕ for as long as $\varepsilon(t) > 0$.

Note that by Theorem 3 T_s is finite. However, in the following we will not use Theorem 3 and prove blowup in finite time together with a characterization of the solution at the singular time. Observe that $\lim_{t \rightarrow T_s} \varepsilon(t) = 0$.

Lemma 4. *Suppose (B) holds on the time interval $[0, T)$, $T > 0$. Then*

$$\eta(x, t) \leq \exp \left(\varepsilon_0^2 K_1 l(t) \int_0^t \varepsilon^{-1}(s) ds \right) \quad (x \in [0, l(t)], 0 \leq t < T).$$

Moreover,

$$(22) \quad \phi(x, t) \leq \kappa x^{1/\beta} \quad (l(t) \leq x < \infty, 0 \leq t < T).$$

Proof. We have $\phi(x, t) \geq \varepsilon(t)$, so $\Phi(z, t) \geq \varepsilon(t)z$. By (20), (18), we have for $0 < s < t$ and $x \in [0, l(t)]$:

$$\eta_t(x, s) \leq \eta(x, s) \varepsilon_0^2 K_1 \frac{l(s)}{\varepsilon(s)}$$

from which the first statement follows.

For all $l(t) \leq x \leq \varepsilon_0^\beta$, let $t_x \leq t$ be the uniquely defined time such that

$$l(t_x) = \varepsilon(t_x)^\beta = x.$$

By the bootstrap assumption, $\phi(x, t_x) \leq \kappa \varepsilon(t_x) = \kappa x^{1/\beta}$. From (12) and the assumption that $g' < 0$ it follows that $\phi(x, t)$ is nonincreasing in t for fixed x . Consequently, $\phi(x, t) \leq \phi(x, t_x) \leq \kappa x^{1/\beta}$ for $t \geq t_x$.

If $x \geq \varepsilon_0^\beta$, then we observe $\phi(x, t) \leq \varepsilon_0 \leq \kappa \varepsilon_0^{1/\beta} \leq \kappa x^{1/\beta}$ by condition (21) and also noting $x \geq \varepsilon_0$. \square

The bootstrap assumption also gives a lower bound on $-\varepsilon_t$. In the following Lemma, the structure of the Biot-Savart law of (3) enters in a crucial way.

Lemma 5. *Suppose (B) holds on $[0, T)$. Then*

$$(23) \quad -\varepsilon_t(t) \geq \frac{\varepsilon_0^2 K_0 c_\beta}{2\kappa} \varepsilon^\beta$$

where $c_\beta = \frac{\beta(\beta+1)}{(2\beta+1)(1-\beta)}$, provided $\varepsilon_0^{1-\beta} \leq 1/2$.

Proof. For $z \geq l(t)$, using the upper bound (22) and (B) yields

$$\begin{aligned}\Phi(z, t) &= \Phi(l(t), t) + \int_{l(t)}^z \phi(\sigma, t) d\sigma \leq \Phi(l(t), t) + \int_{l(t)}^z \kappa \sigma^{\frac{1}{\beta}} d\sigma \\ &\leq \Phi(l(t), t) + \frac{\kappa\beta}{\beta+1} (z^{\frac{1}{\beta}+1} - l^{\frac{1}{\beta}+1}(t)) \leq \kappa\varepsilon(t)l(t) + \frac{\kappa\beta}{\beta+1} (z^{\frac{1}{\beta}+1} - l^{\frac{1}{\beta}+1}(t)) \\ &\leq \kappa \left(l^{\frac{1}{\beta}+1}(t) + \frac{\beta}{\beta+1} z^{\frac{1}{\beta}+1} \right).\end{aligned}$$

Using this and (18)

$$\begin{aligned}\int_0^\infty \frac{-g'(z) dz}{\Phi(z, t)} &\geq \frac{\varepsilon_0^2 K_0}{\kappa} \int_{l(t)}^1 \frac{z dz}{l^{1+\frac{1}{\beta}}(t) + \frac{\beta}{1+\beta} z^{1+\frac{1}{\beta}}} \geq \frac{\varepsilon_0^2 K_0}{\kappa} \int_{l(t)}^1 \frac{z dz}{(1 + \frac{\beta}{1+\beta}) z^{1+\frac{1}{\beta}}} \\ &= \frac{\varepsilon_0^2 K_0}{\kappa} \left(1 + \frac{\beta}{1+\beta} \right)^{-1} \frac{\beta}{1-\beta} (l^{1-\frac{1}{\beta}}(t) - 1) \\ &= \frac{\varepsilon_0^2 K_0 c_\beta}{\kappa} (\varepsilon^{\beta-1}(t) - 1).\end{aligned}$$

Thus

$$-\varepsilon_t(t) \geq \varepsilon(t) \int_0^\infty \frac{-g'(z) dz}{\Phi(z, t)} \geq \frac{\varepsilon_0^2 K_0 c_\beta}{\kappa} \varepsilon^\beta(t) (1 - \varepsilon^{1-\beta}(t)) \geq \frac{\varepsilon_0^2 K_0 c_\beta}{2\kappa} \varepsilon^\beta(t)$$

provided $\varepsilon_0^{1-\beta} \leq 1/2$. □

Lemma 6. *Suppose the following three conditions hold:*

$$\begin{aligned}(24) \quad &\varepsilon_0 \leq \kappa \varepsilon_0^{1/\beta}, \\ &\varepsilon_0^{1-\beta} \leq 1/2, \\ &\frac{2K_1}{K_0 c_\beta \beta} < \frac{\log \kappa}{\kappa}.\end{aligned}$$

Then (B) holds on the whole interval $[0, T_s)$.

Proof. Observe first that (B) holds for $\phi(\cdot, 0)$ since $\kappa > 1$. Because of continuity in time, there exists a small time interval $[0, \delta)$ in which (B) holds. Suppose (B) does not hold on the whole interval $[0, T_s)$, and let

$$T := \sup\{t \in [0, T_s) : (B) \text{ holds on } [0, t]\} < T_s.$$

Observe that (B) holds on $[0, T)$ and that the monotonicity of ϕ implies that

$$(25) \quad \phi(l(T), T) = \kappa\varepsilon(T).$$

The first two conditions of (24) allow us to use Lemma 5 to estimate

$$\begin{aligned} \varepsilon_0^2 K_1 l(T) \int_0^T \frac{ds}{\varepsilon(s)} &= \varepsilon_0^2 K_1 l(T) \int_0^T \frac{-\varepsilon_t(s)}{-\varepsilon_t(s)\varepsilon(s)} ds \leq \frac{2K_1\kappa}{K_0 c_\beta} l(T) \int_0^T \frac{-\varepsilon_t(s)}{\varepsilon^{1+\beta}(s)} ds \\ &\leq \frac{2K_1\kappa}{K_0 c_\beta \beta} l(T) (\varepsilon^{-\beta}(T) - \varepsilon_0^{-\beta}) \leq \frac{2K_1\kappa}{K_0 c_\beta \beta} (1 - (\varepsilon(T)/\varepsilon_0)^\beta) \\ &\leq \frac{2K_1\kappa}{K_0 c_\beta \beta}. \end{aligned}$$

Thus by Lemma 4,

$$\eta(l(T), T) \leq \exp\left(\frac{2K_1\kappa}{K_0 c_\beta \beta}\right).$$

Now, using the third line of (24),

$$\phi(l(T), T) \leq \varepsilon(T) \eta(l(T), T) \leq \varepsilon(T) \exp\left(\frac{2K_1\kappa}{K_0 c_\beta \beta}\right) < \kappa \varepsilon(T),$$

contradicting (25). \square

Conclusion of the proof of Theorem 2. We show that it is possible to choose $0 < \beta < 1, 0 < \varepsilon_0 < 1, \kappa > 0$ such that (24) are satisfied. This is done as follows: Fix $\kappa > 2$ and observe that the first and second condition of (24) together are equivalent to

$$(26) \quad -\frac{\beta}{1-\beta} \log \kappa \leq \log \varepsilon_0 \leq -\frac{\log 2}{1-\beta}.$$

(26) holds for sufficiently small ε_0 provided β can be chosen such that $-\frac{\beta}{1-\beta} \log \kappa < -\frac{\log 2}{1-\beta}$, i.e.

$$(27) \quad \log \kappa > \frac{\log 2}{\beta}.$$

Since $\kappa > 2$, (27) holds if β is close enough to 1. Also the third condition of (24) holds if β is close enough to 1, which follows from $c_\beta \rightarrow \infty$ as $\beta \rightarrow 1$.

Taking into account (24), we can now apply Lemma 6 to see that (B) holds on the whole existence interval $[0, T_s)$ of the solution. (23) now implies that

$$\varepsilon_t(t) \leq -C\varepsilon^\beta(t)$$

for some positive, fixed $C > 0$, and hence T_s must be finite and $l(T_s) = \varepsilon^\beta(T_s) = 0$.

We show now that θ defined by (9) is regular outside the origin, even at time $t = T_s$. Note that $\phi(x, t)$ for fixed x is monotone nonincreasing in t and $\phi(x, t) \in [0, \varepsilon_0]$ for $t < T_s$. Hence the pointwise limit

$$\phi(x, t) = \lim_{t \rightarrow T_s} \phi(x, t)$$

exists and is ≥ 0 everywhere. Let $s_0 \geq 0$ be the infimum of all numbers $x > 0$ with the property that $\phi(x, T_s) > 0$ (see Figure 2). Note that the set over which the infimum is taken

is not empty, since $\phi(x, t) = \varepsilon_0 > 0$ for all $0 \leq t < T_s$ if x is outside the support of g' . Moreover,

$$\Phi(x, t) \geq \varepsilon_0(x - a) \quad (x \geq a, 0 \leq t < T_s)$$

if $\text{supp } g' = [0, a]$.

From $\Phi(x, t) = \int_0^x \phi(\sigma, t) d\sigma$, we see that $\Phi(x, t)$ is nonincreasing in time for fixed x and $\Phi(x, T_s) \geq 0$. Consequently, the pointwise limit $\Phi(x, T_s)$ exists, and

$$\Phi(x, T_s) = 0 \text{ for } 0 \leq x \leq s_0, \text{ and } \Phi(x, T_s) > 0 \text{ for all } x > s_0.$$

Observe that $\Phi(x, T_s)$ is continuous and strictly increasing in x for $x \geq s_0$ since $\phi(x, T_s) > 0$ for $x > s_0$. An elementary argument (using e.g. the fact that $\Phi(x, t)$ is nonincreasing) proves that as $t \rightarrow T_s$, $\Phi^{-1}(y, t)$ converges for all $y > 0$ to a limit $\Phi^{-1}(y, T_s)$. The function $\Phi^{-1}(\cdot, T_s)$ is the inverse of $\Phi(\cdot, T_s)$ restricted to the interval (s_0, ∞) . (9) shows that the pointwise limit $\theta(y, T_s)$ exists for all $y > 0$. By Lemma 7 below, $\phi(\cdot, T_s) \in C^1(s_0, \infty)$ and so again by (10) $\theta(\cdot, T_s) \in C^2(0, \infty)$. This proves that θ remains regular outside the origin even at the singular time T_s .

Finally, we prove the bound (6). First we look at the case $s_0 = 0$. A key observation is that from (22) we get the upper bound

$$\Phi(z, T_s) \leq C(\beta, \kappa) z^{\frac{1}{\beta}+1} \quad (0 \leq z < \infty),$$

implying

$$(28) \quad \Phi^{-1}(y, T_s) \geq \tilde{C}(\beta, \kappa) y^{\frac{\beta}{\beta+1}}.$$

From (10) we get, using the substitution $\Phi^{-1}(\tilde{y}, t) = z$,

$$\theta(y, t) = \theta(0, t) + \int_0^y \frac{g'(\Phi^{-1}(\tilde{y}, t))}{\phi(\Phi^{-1}(\tilde{y}, t))} d\tilde{y} = \theta_0(0) + \int_0^{\Phi^{-1}(y, t)} g'(z) dz$$

Now pass to the limit $t \rightarrow T_s$ and continue the estimate by using first (18) and then (28),

$$\int_0^{\Phi^{-1}(y, T_s)} g'(z) dz \leq -C(\varepsilon_0, K_0, \beta, \kappa) y^{\frac{2\beta}{\beta+1}}$$

for all $y \in [0, \Phi(1, T_s)]$. Note that $\Phi(1, T_s) > 0$ because of $s_0 = 0$.

Since $\theta(y, T_s) \leq \theta(\Phi(1, T_s), T_s)$ for $y \geq \Phi(1, T_s)$ we can define $\nu = \frac{2\beta}{\beta+1} < 1$ and adjust the constant such that (6) holds (note that $\theta(y, T_s)$ has compact support).

If $s_0 > 0$, we note $\sup_{\mathbb{R}_+} \theta(y, T_s) < \theta(0, T_s) = \theta_0(0)$. Hence, the constant c can be adjusted such that (6) holds. This concludes the proof of Theorem 2.

Lemma 7. *Suppose $\lim_{t \rightarrow T_s} \Phi(x_0, t) > 0$. Then $\phi(\cdot, T_s) \in C^1[x_0, \infty)$.*

Proof. Since for fixed t , $\Phi(x, t)$ is non-decreasing in x , $\Phi(x, t) \geq \Phi(x_0, t) > 0$ for all $x \geq x_0$, $t < T_s$. Using this lower bound for Φ , the C^1 -norm of the right-hand-side of equation (12) on the interval $[x_0, \infty)$ can be bounded:

$$\|\phi_t(\cdot, t)\|_{C^1[x_0, \infty)} \leq C(x_0)$$

for all $t < T_s$, where $C(x_0)$ depends on x_0 but not on t . This implies the convergence $\phi(\cdot, t) \rightarrow \phi(\cdot, T_s)$ in $C_1([x_0, \infty))$ as $t \rightarrow T_s$. \square

Remark 2. *It is not necessary to choose $(\varepsilon_0, g(\cdot; \varepsilon_0))$ with sufficiently small ε_0 for the pair (ϕ_0, g) . In fact, we could have worked with the canonical choice $(1, \theta_0)$. In this case, one first proves a bound of the form $-\varepsilon_t \leq k\varepsilon$ with $k > 0$. This means that after some positive time T_0 , $\varepsilon(t)$ is small enough such that (24) can be satisfied.*

Remark 3. *For the moment, we leave the question open if a needle-like discontinuity can really form, or if generically a cusp is obtained at the singular time. In this context, it is interesting to observe that a suitable lower bound*

$$(29) \quad \phi(x, T_s) \geq c_0 x^p$$

with $c_0 > 0$, $p > 1$ would suffice to exclude the needle scenario, and would nicely complement (22). A lower bound for $\phi(\cdot, t)$ for all $t < T_s$ was obtained by A. Zlatoš [17], however, his lower bound does not give any information in the limit $t \rightarrow T_s$.

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